SELF-SIMILAR MOVEMENT OF A VISCOUS GAS IN A CHANNEL

M. A. Gol'dshtik

UDC 532.517.2

The results presented in [1] refer primarily to dropping liquids for which the influence exerted by the thermal conditions on the flow is related to the temperature dependence of the viscosity. The self-similar flow of a viscous gas in a channel with a linearly increasing wall temperature is examined in this paper. The influence exerted by the Reynolds and Prandtl numbers on heat exchange and the hydrodynamics of the flow is analyzed.

§ 1. Let us consider an idealized case in which the kinematic viscosity ν and the coefficient of thermal conductivity $\varkappa$ are treated as constant and the density ρ is related to the temperature T in the following dependence

$$\rho = \rho_0 T_0/T,$$

which is used by Boussinesq as a rough approximation of the equation $p = \rho RT$ for low values of $\Delta p/p$.

Given these assumptions the problem of the stationary flow of a viscous gas in a plane channel with a linearly increasing wall temperature, since $T_W = T_0 x$, is capable of a self-similar solution in the form

$$v_x = v_0 u(y)x;\ v_y = 0;\ T = T_0\theta(y)x. \tag{1.1}$$

The equations of movement and energy for this form of solution can be written as

$$v_x \partial v_x/\partial x = -(1/\rho)\partial p/\partial x + \nu \partial^2 v_x/\partial y^2;$$
$$v_x \partial T/\partial x = \varkappa \partial^2 T/\partial y^2, \tag{1.2}$$

where x and y are Cartesian coordinates; v_x and v_y are the longitudinal and transverse components of the gas velocity; p is pressure; ρ_0 and T_0 are certain scale values of the density and temperature; and v_0 is the mean gas flow rate assumed to be expressed by

$$v_0 = \frac{1}{\rho_0}\int_0^1 \rho v_x dy. \tag{1.3}$$

The half-width of the channel h is taken as the scale of length.

By inserting relation (1.1) into Eq. (1.2), we obtain

$$u'' = \mathrm{Re}\,(u^2 - \chi a^2\theta); \tag{1.4}$$

$$\theta'' = \mathrm{Re}\,\sigma u\theta, \tag{1.5}$$

where the prime indicates differentiation in terms of y; $\mathrm{Re} = v_0 h/\nu$ is the Reynolds number; $\sigma = \nu/\varkappa$ is the Prandtl number; $a^2 = (h/\rho_0 v_0^2)\,|dp/dx|$ is the coefficient of resistance; and $\chi = \pm 1$. In expression (1.4) the χ value $\chi = +1$ corresponds to a movement of the gas in a positive direction when it is heated. When it moves in the opposite direction the sign of the pressure gradient is governed by two opposing factors: the friction between the flow and the wall and the deceleration of the gas in the direction in which it is moving, so that a case corresponding to a value of $\chi = -1$ is possible (friction predominating).

By assuming the flow to be symmetrical about the axis of the channel (y = 0) the following boundary conditions can be imposed:

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 47–54, May–June, 1976. Original article submitted May 30, 1975.

$$u'(0) = \theta'(0) = 0;\ u(1) = 0;\ \theta(1) = 1.$$

In addition, the following relation follows from (1.3):

$$\int_0^1 \frac{u}{\theta}\, dy = 1, \tag{1.6}$$

which is used to determine the unknown parameter a in (1.4).

The nonlinear boundary-value problem being posed is solved by replacing variables,

$$u = aw,\ y = z/\sqrt{a\,\mathrm{Re}}, \tag{1.7}$$

which gives Eqs. (1.4) and (1.5) the forms

$$w'' = w^2 - \chi\theta; \tag{1.8}$$

$$\theta'' = \sigma w\theta. \tag{1.9}$$

It is useful in solving this derived set of equations, to examine the auxiliary Cauchy problem

$$w(0) = w_0;\ w'(0) = 0;\ \theta(0) = \theta_0;\ \theta'(0) = 0. \tag{1.10}$$

In relations (1.8)-(1.10) the prime indicates differentiation in terms of the variable z.

The problem as posed contains the two free parameters w_0 and θ_0, which can be determined by applying the following considerations. Let the magnitude of θ_0 be fixed. Then, as shown below, the point z_0 is found for certain values of w_0, where $w(z_0) = 0$. If, by varying the parameter w_0, the relation $\theta(1) = 1$ can be completed, the original boundary conditions will be satisfied when $y = 1$, from which according to (1.7) it can be assumed that

$$z_0 = \sqrt{a\,\mathrm{Re}}. \tag{1.11}$$

The relation (1.6) in the new variables takes the form

$$\frac{a}{\sqrt{a\mathrm{Re}}} \int_0^{z_0} \frac{w}{\theta}\, dz = 1. \tag{1.12}$$

The magnitudes of a and Re are determined by using formulas (1.11) and (1.12):

$$a = z_0 \left(\int_0^{z_0} \frac{w}{\theta}\, dz \right)^{-1};\ \mathrm{Re} = z_0 \int_0^{z_0} \frac{w}{\theta}\, dz. \tag{1.13}$$

The magnitude of θ_0 for a given σ can be treated as the basic parameter of the problem with one or more Re values corresponding to each of its values. The coefficient of friction c_f and the Nusselt criteria Nu can be determined from the relations

$$\begin{gathered} c_f = -(\nu/v_0^2)\, \partial v_x/\partial y|_{y=1} = -a\sqrt{(a/\mathrm{Re})}\, w'(z_0) = cx; \\ \mathrm{Nu} = \alpha h/\lambda = [1/(T_w - T_0)]\partial T/dy|_{y=1} = [1/(1-\theta_0)]d\theta/dy|_{y=1} = z_0\theta'(z_0)/(1-\theta_0). \end{gathered} \tag{1.14}$$

§ 2. When $\sigma = 0$ the statement of the problem must be altered slightly, since, according to (1.9), $\theta \equiv 1$ and it is not possible to satisfy the condition $\theta(1) = 1$ by selecting values for w_0. In this case the following problem requires solution:

$$w'' = w^2 - \chi;\ w(0) = w_0;\ w'(0) = 0. \tag{2.1}$$

The correlations in (1.13) still remain valid since, the connection between w_0 and Re can be established directly.

The multiplication of (2.1) by w' followed by integration gives the following relation:

$$w'^2 = \frac{2}{3}(w^3 - w_0^3) - 2\chi(w - w_0) \equiv F(w). \tag{2.2}$$

Here the boundary condition $w'(0) = 0$ is taken into account. From (2.2) we obtain $w' = \sqrt{F(w)}$, from which it follows that

$$z = \pm \int_{w_0}^{w} \frac{dw}{\sqrt{F(w)}}. \tag{2.3}$$

Consequently, the magnitude of z_0 governed by the condition $w(z_0) = 0$ can be written as

$$z_0 = \pm \int_{w_0}^{0} \frac{dw}{\sqrt{F(w)}}. \tag{2.4}$$

In expressions (2.3) and (2.4) the sign should be selected on the basis of the condition of positiveness of z_0 so that the sign is the opposite of the sign of the w_0. In the case of $\chi = -1$ the problem (2.1) has only an incremental solution with a positive curvature so that only the negative values of w_0 can correspond to the original boundary-value problem. A minus sign should be assigned to the value of the root in (2.3). The w(z) function has a single root z_0 and throughout the $0 \leq z \leq z_0$ range $w(z) \leq 0$ so that, according to (1.13), (1.7), and (1.1), $a < 0$; $Re < 0$; $u \geq 0$; $v_0 < 0$; and $v_x \leq 0$, which characterizes movement in the negative direction.

After replacing the variables $w = w_0 t$ the expression (2.4) can be recorded as

$$z_0 = \sqrt{-w_0} \int_0^1 \frac{dt}{\Delta}; \quad \Delta = \sqrt{2/3 w_0^2 (1 - t^3) - 2\chi (1 - t)}\,. \tag{2.5}$$

When $|w_0| \ll 1$, according to (2.5), $z_0 \sim (-w_0)^{1/2}$ and, when $|w_0| \gg 1$, $z_0 \sim (-w_0)^{-1/2}$ and there is thus a maximum value of z_0 for any w_0. By using (1.13) and (2.5), it can be written

$$Re = -w_0^2 \int_0^1 \frac{dt}{\Delta} \int_0^1 \frac{t\,dt}{\Delta}.$$

From this it can be seen that when $|w_0| \ll 1$, $Re \sim w_0^2$, and when $|w_0| \to \infty$, Re tends to a certain finite number equal, according to calculations, to -1.814. Thus, with a positive pressure gradient an inverse movement with high Reynolds numbers is impossible.

In the case of $\chi = +1$ the F(w) function, as determined by expression (2.2), is best represented as

$$F(w) = 2/3(w - w_0)(w - w_1)(w - w_2), \tag{2.6}$$

where

$$w_{1,2} = 1/2\left(\pm \sqrt{12 - 3w_0^2} - w_0\right).$$

If w(z) is the solution to the original boundary-value problem then $w(z_0) = 0$, so that $F(0) = 2w_0(1 - 1/3w_0^2)$. Since by definition $F(w) \geq 0$, only those values of w_0 which satisfy the inequalities

$$0 \leqslant w_0 \leqslant \sqrt{3} \quad \text{or} \quad w_0 \leqslant -\sqrt{3}$$

are admissible.

In the $0 \leq w_0 \leq \sqrt{3}$ and $-2 \leq w_0 \leq -\sqrt{3}$ ranges the w_1 and w_2 roots are real and when $w_0 < -2$ they are complex. The function (2.6) is a cubic parabola with $F \to \pm\infty$ when $w \to \pm\infty$. If $w_0 < -2$, then F(w) has a single real root w_0 and w(z) increases smoothly up to infinity. This follows from (2.1) and (2.2). When $w_0 \to \infty$, $Re \to -1.814$ as for $\chi = -1$. When $w_0 = -2$, w(z) increases smoothly and tends asymptotically to a value of $w_1 = w_2 = 1$. If $-2 < w_0 \leq -3$, then w(z) increases while the F(z) function does not reach the subsequent zero $0 \leq w_1 < 1$, where w exhibits a maximum. Further, as z rises w(z) falls, which corresponds to movement backward along the phase trajectory in the (F, w) plane. In this case the w function is periodic, oscillating between values of w_0 and w_1 with a period

$$\xi = \sqrt{\frac{3}{2}} \int_{w_0}^{w_1} \frac{dw}{\sqrt{(w - w_0)(w - w_1)(w - w_2)}}. \tag{2.7}$$

When $w_0 = -\sqrt{3}$, $w_1 = 0$, so that w(z) reaches a maximum when $w = 0$ and oscillates between values of $-\sqrt{3}$ and 0 with a period $\xi = (3/4)^{1/4}(2\sqrt{2}\pi)^{-1}\Gamma^2(1/4) \simeq 2.45$ corresponding to a Reynolds numbers of $Re = -3/2\pi = -4.71$. In the $-\sqrt{3} < w_0 < 0$ range the w(z) function is periodic and negative so that the original boundary-value problem has, as already noted, no solution, i.e., the backward movement of the gas is not possible when the Re numbers are fairly high in terms of the modulus.

When $0 < w_0 < 1$, the w(z) function is also periodic with a period of (2.7) and it can be found by shifting the solutions corresponding to values of $-2 < w_0 \leq -\sqrt{3}$ onto half the period. The minima of the w(z) function then lie in the $(-2; -\sqrt{3})$ range; in other words, there is a finite value of z_0 for each w_0 in the range under consideration.

For the periodic solutions the magnitude z_0 can taken to be the first root of the $w(z)$ function. Self-similar periodic solutions governed by the input sign-variable velocity distribution exist but they are not examined here.

In the case of $w_0=0$, Eq. (2.2) is homogeneous and therefore $w(z)\equiv 0$. If it is considered that for low values of w_0, $w(z)$ is small throughout the $(0, z_0)$ range, then according to (2.1) it should be assumed that $w'' = -1$:

$$w = w_0 = 1/2z^2;\ z_0 = \sqrt{2w_0};\ \mathrm{Re} = 4/3w_0^2;\ a = 3/2w_0.$$

Hence, according to (1.7) a Poiseuille parabola is obtained:

$$u = 3/2(1 - y^2),$$

with the coefficients a and c being governed by the relation

$$a = \sqrt{3/\mathrm{Re}},\ c = a^2 = 3/\mathrm{Re}.$$

In the other limiting case when $w_0=1$, $w_1=1$; $w_2=-2$;

$$z = \infty;\ F(w) = 2/3(1 - w)^2(2 + w);$$
$$w' = -(1 - w)\sqrt{2/3(2 + w)};\ w(0) = 1.$$

The latter equation has a trivial solution $w\equiv 1$ which cannot be used to derive a solution to the original boundary-value problem. It is therefore assumed that $w_0=1-\varepsilon$, where $\varepsilon \ll 1$.

It can be seen that $F(0)=4/3(1-1/2\varepsilon^2-1/2\varepsilon^3)$. Consequently, correct within ε-linear terms,

$$F(w) = 2/3(1 - w)^2(2 + w).$$

By inserting this expression into (2.2) and integrating when $w(0)=1-\varepsilon$, we obtain

$$z = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{3} + \sqrt{2 - w}}{\sqrt{3} - \sqrt{2 + w}} \frac{1 - \sqrt{1 - 1/3\varepsilon}}{1 + \sqrt{1 - 1/3\varepsilon}}.$$

If it assumed that $w=0$, then retaining the terms of the order of ε

$$z_0 \simeq (1/\sqrt{2}) \ln (1.21/\varepsilon).$$

Further, it can be found that

$$\int_0^{z_0} w\,dz = \int_0^{w_0} z' w\,dw = \int_0^{w_0} \frac{w\,dw}{\sqrt{F(w)}} \simeq z_0 - 0.777.$$

Taking this result into account, according to (1.13),

$$a = 1 + 0.778/z_0;\ \mathrm{Re} = z_0(z_0 - 0.777).$$

From the last formula it is clear that low ε correspond to high Re values. The asymptotic ε (Re) relationship takes the form

$$\varepsilon = 0.697 \exp(-\sqrt{2\,\mathrm{Re}}).$$

Using this correlation it can be established that

$$a \simeq 1;\ z_0 \simeq \sqrt{\mathrm{Re}};\ w'(z_0) \simeq -2\sqrt{3};\ c \simeq 2/\sqrt{3\,\mathrm{Re}}.$$

It can be concluded from these results that in the area near the walls given high Reynolds numbers the solution found is of the nature of a laminar boundary layer and the velocity at the core of the flow is virtually constant. This rearrangement of the profile compared with the isothermic case takes place at a constant cross-sectional temperature and is due solely to the axial acceleration of the flow which generates a significant increase in the frictional resistance compared with the Poiseuille flow, for which $c=3/\mathrm{Re}$, as well as an increase in the total drag a^2.

In general, the solution to Eq. (2.2) is expressed as an elliptical integral:

$$z = \int_w^{w_0} \frac{dw}{\sqrt{F(w)}}.$$

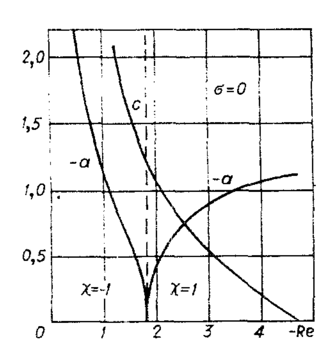

Fig. 1

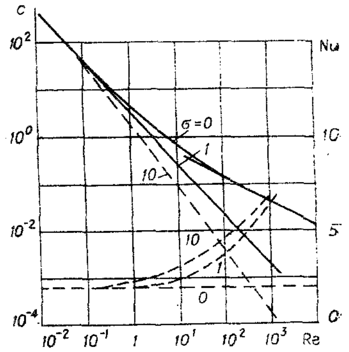

Fig. 2

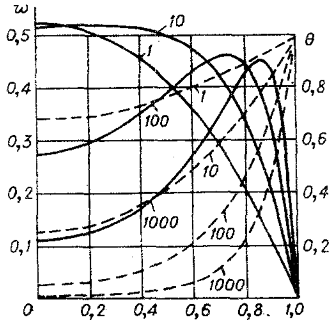

Fig. 3

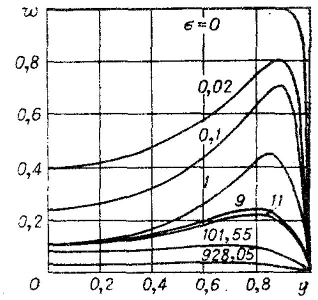

Fig. 4

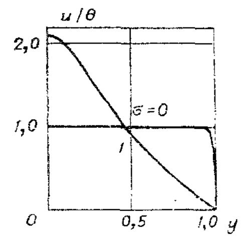

Fig. 5

Thus, w(z) can be written as an elliptical function. It is, however, easier to solve this problem numerically for different w_0. The results of such calculations are shown in Fig. 1 for Re < 0 and in Fig. 2 for Re > 0.

In the case of low Reynolds numbers the problem can be solved for any values of σ.

The solution to Eqs. (1.4) and (1.5) is derived in the form of an expansion:

$$u = u_1 \operatorname{Re} + u_2 \operatorname{Re}^2 + \ldots; \quad \theta = 1 + \theta_1 \operatorname{Re} + \theta_2 \operatorname{Re}^2 + \ldots.$$

For the leading coefficients of the expansion a set of equations is derived:

$$u_1'' = -\chi a^2; \quad \theta_1'' = 0; \quad \theta_2'' = \sigma u_1,$$

for which the following boundary conditions are imposed:

$$u_1'(0) = u_1(1) = \theta_1'(0) = \theta_1(1) = \theta_2'(0) = \theta_2(1) = 0.$$

The solution takes the form

$$u_1 = (\chi a^2/2)(1 - y^2); \quad \theta_1 = 0; \quad \theta_2 = (\sigma\chi a^2/24)(6y^2 - y^4 - 5).$$

Consequently,

$$\theta = 1 + (\sigma\chi a^2/24)\operatorname{Re}^2 (6y^2 - y^4 - 5); \quad \theta_0 = 1 - (5/24)\sigma\chi a^2 \operatorname{Re}^2.$$

By inserting these results into (1.14) we obtain

$$\mathrm{Nu} = 1.6. \quad (2.8)$$

It can be shown that even when $\sigma \to 0$ the asymptotic relationship (2.8) occurs.

The results of a numerical solution of the problem in the form of c(Re) and Nu(Re) relationships are shown in Fig. 2 by a dashed line for three values of the parameter $\sigma = 0$, 1, and 10. The most characteristic property of the velocity profiles when $\sigma > 0$ is the lack of smoothness in the distribution of velocities through the channel cross section for high Reynolds numbers [Fig. 3, in which for $\sigma = 1$ the w(y) relationships for different Re numbers are shown by solid lines and the temperature distribution θ(Re) by dotted lines]. The Reynolds numbers Re = 1, 10, 100, and 1,000 correspond to values of $a = 2.05$, 0.861, 0.361, and 0.148. The set of w(y) profiles for Re = 1,000 and different Prandtl numbers ($0 < \sigma < 1{,}000$) are shown in Fig. 4. The profiles of the mass velocity $\rho v_x \sim u/\theta$ are smooth (Fig. 5, in which u/θ profiles for $\sigma = 0$ and $\sigma = 1$ are compared at Re = 1,000).

Thus, the unevenness in the density distribution through the channel cross section generates a reduction in the volumetric velocity and an increase in the mass velocity in the area around the axis.

The author wishes to thank A. F. Seleznev for carrying out the calculations and V. N. Shtern for discussing the paper.

LITERATURE CITED

1. B. S. Petukhov, Heating Exchange and Resistance Accompanying the Laminar Flow of a Fluid in Pipes [in Russian], Énergiya, Moscow (1967).

DETERMINING THE RADIUS OF THE AIR VORTEX DURING THE LAMINAR FLOW OF A LIQUID IN A CENTRIFUGAL ATOMIZER

Yu. Z. Nekhamkin, B. D. Strelkov, and Yu. I. Khavkin

UDC 532.517

In existing theories of centrifugal atomizers, such as that of Abramovich [1], in order to determine the radius r_0 of the air vortex the conditions of the maximum rate of flow or some other extremal principle are conventionally employed. In this paper the radius of the air vortex will be determined from the equations of motion of a viscous incompressible liquid.

The atomizer under consideration is illustrated schematically in Fig. 1. Phenomena taking place in the boundary layers close to the ends are not taken into account. The region of flow is divided into two zones.

All the quantities in this paper are dimensionless; lengths are given in terms of the radius of the outlet nozzle r_1, and velocities, in terms of the velocity in the inlet channels V.

In zone I($1 \le r \le a$) the flow is quite flat, of the vortical sink type, i.e., $v = v(r)$, $u = 0$, $w = w(r)$, where v is the radial velocity component, u is the axial component, and w is the circumferential component.

Equations for the velocity components in zone I were obtained in [2]:

$$v = -\varkappa/r; \quad w = C_1 r^{1-\varkappa \mathrm{Re}} + C_2/r,$$

where $\varkappa = f/2\pi L r_1$; $\mathrm{Re} = V r_1/\nu$; f is the cross-sectional area of the inlet channels.

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 54-57, May-June, 1976. Original article submitted April 8, 1975.